Summarizing data

EDS 222

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Fall 2023

Today

Types of variables

- Categorical, numerical, ordinal, …

Probability density functions

- Definitions, the normal pdf, skew

Summary statistics

- Central tendency and spread, quantiles, outliers

Law of large numbers

- How big does my sample need to be?

Assignment #1 check-in: How's it going?

Reminder: OH Thursdays, Pine Room, 3:30-4:30pm

Types of variables

Types of variables

Numerical variables

> Object class `numeric` in `R`
>
> - Can take on a wide range of possible values
> - Makes sense to add, subtract, multiply, etc.

- Examples:
    - Height of the tree canopy across the Amazon
    - Length of Atlantic swordfish
    - Daily average temperature

Discrete numerical variables take on only a limited set of values, often counts (e.g., population)

Continuous numerical variables: can take on infinite values within a range (e.g., arsenic concentration in groundwater)

Types of variables

Numerical variables

Source: Allison Horst

Types of variables

Categorical variables

> Object class `factor` in `R`

- Values correspond to one of a fixed number of categories
- Possible values are called **levels**
- Examples:
    - Land use type
    - Species of tree
    - Age group (e.g., <15, 15-64, 65+) (watch out! continuous numerical data can often be stored as a categorical variable!)

Types of variables

Categorical variables

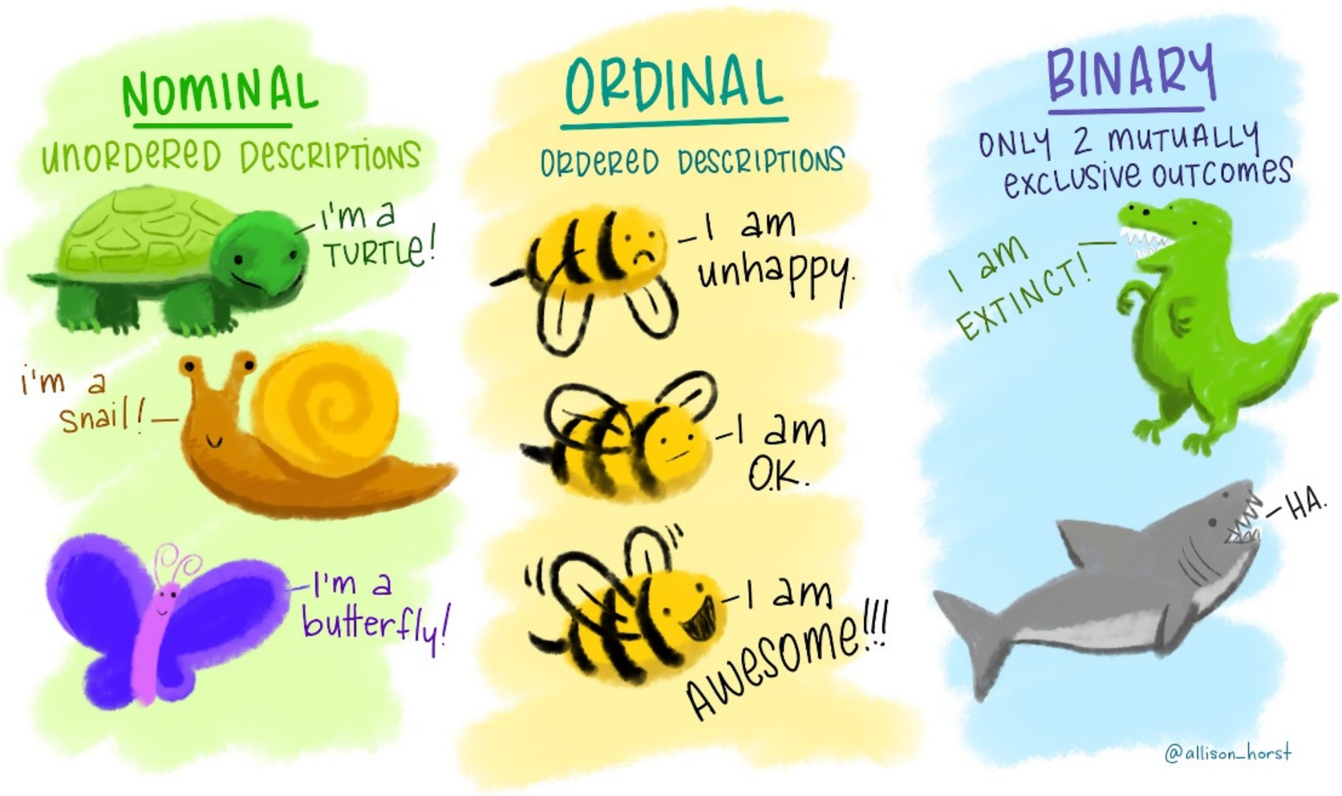
Nominal variables are unordered descriptions

Ordinal variables are categories with a natural ordering

Binary variables only take on 0 or 1

Types of variables

Categorical variables



Source: Allison Horst

Probability density functions

Probability density functions

Remember: when we do statistics, we use *statistics* from a sample to learn about *parameters* of a population.

A **variable** is a representation of something we care about in a population (e.g., nitrate concentration of groundwater).

Many parameters we care about tell us something about what values we might see for our variable in the population (e.g., average nitrate concentrations).

Probability density functions are mathematical functions that tell us: how likely are we to see values of a given range?

Probability density functions

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Probability density functions

For *continuous* variables, the **probability density function (p.d.f.)** tells us the probability that a variable falls within a given range of values.

Formally: The **p.d.f.** of a continuous variable X with support (i.e., range of possible values) S is an integrable function $f(x)$ satisfying:

1. $f(x)$ is positive for all x in S
2. The area under the curve $f(x)$ over the entire support S is equal to 1:

$$\int_S f(x)dx = 1$$

3. The probability that x falls between A and B is:

$$Pr(A \leq x \leq B) = \int_A^B f(x)dx$$

Why isn't this simpler?

> Q: Why can't I just interpret $f(x)$ as the probability that $X = x$?
>
> A: Because continuous variables have ∞ possible values...the probability that your variable X exactly equals x is zero!

Luckily, for **discrete variables** it *is* this simple!

For *discrete* variable x , the **probability mass function (p.m.f.)** $f(x)$ tells us the probability that $X = x$.

Formally: The **p.m.f.** of a discrete variable X with support (i.e., range of possible values) S is a function $f(x)$ satisfying:

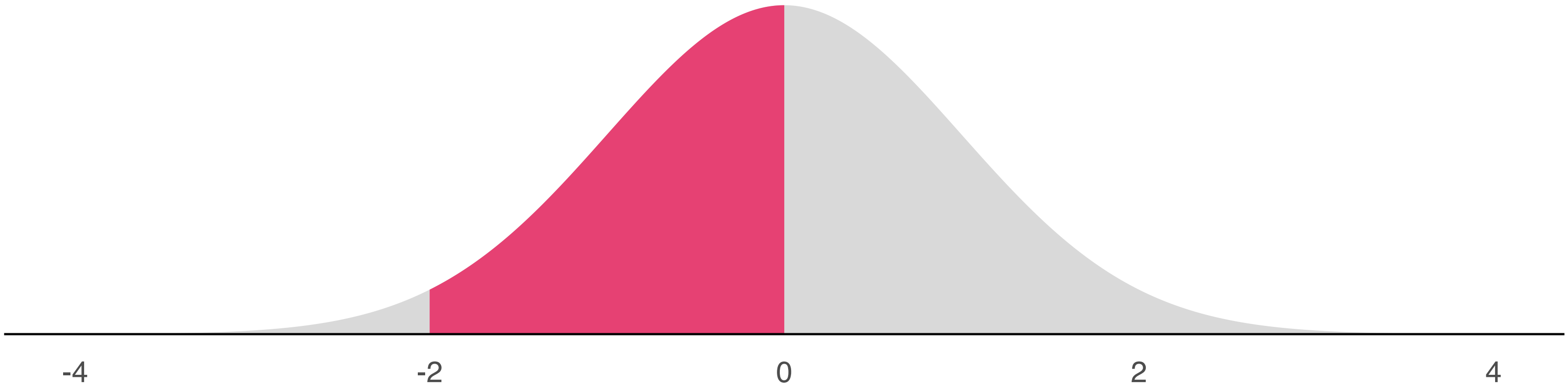
1. $P(X = x) = f(x) > 0$ for all x in support S
2. $\sum_{x \in S} f(x) = 1$
3. $P(A \leq x \leq B) = \sum_{x=A}^{x=B} f(x)$

Probability density functions (visual)

P.d.f.'s help us characterize the distribution of our population. The most common/famous ones get names (e.g., normal, Gamma, t,...)

Let's look at a **normal** distribution*

The probability this normally distributed variable takes on a value between -2 and 0 is shown in pink:

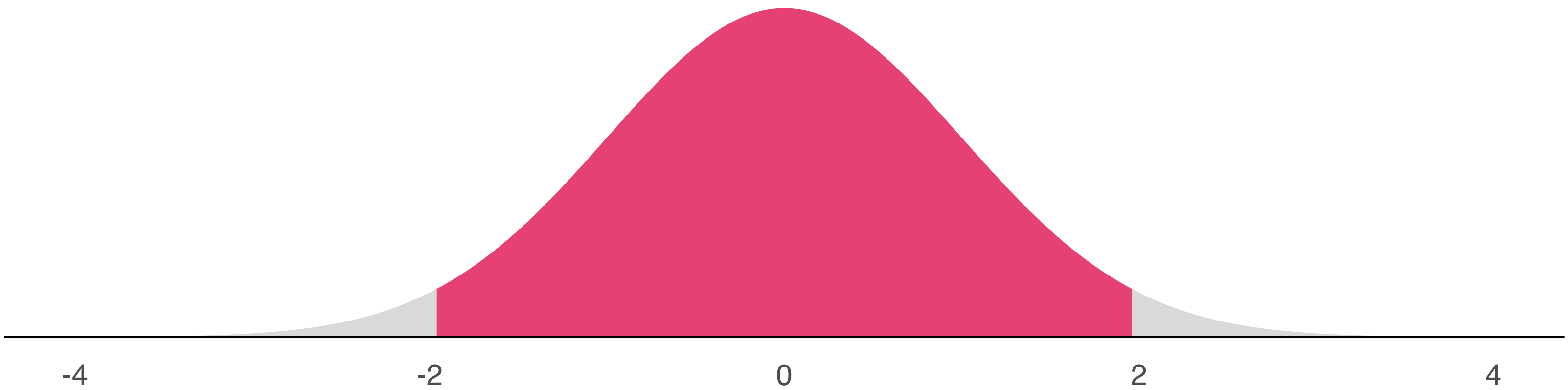


*This distribution happens to be what's called "standard" normal. We'll get into the weeds later!

Probability density functions (visual)

Let's look at a **normal** distribution*

The probability this normally distributed variable takes on a value between -2 and 2 is shown in pink:



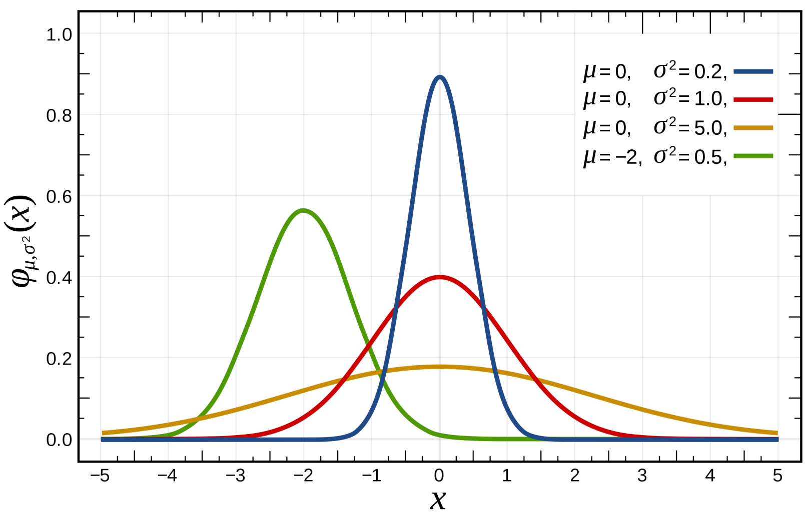
*Yep, still a "standard" normal. Details later.

The normal distribution

There are infinite different normal distributions. They all have the following p.d.f.:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)}$$

where μ is the mean (i.e., average) and σ is the standard deviation (will define soon). μ and σ are **parameters** describing the population p.d.f.



Shapes of probability distributions

Key terms to describe p.d.f.'s:

1. A distribution can have **skew** (e.g., log-normal)
2. A distribution can have a long **right tail** or **left tail** (e.g., fat-tailed climate sensitivity distributions!)
3. A distribution can be **symmetric**
4. A distribution can be **unimodal**, **bimodal**, or **multimodal**

Shapes of probability distributions

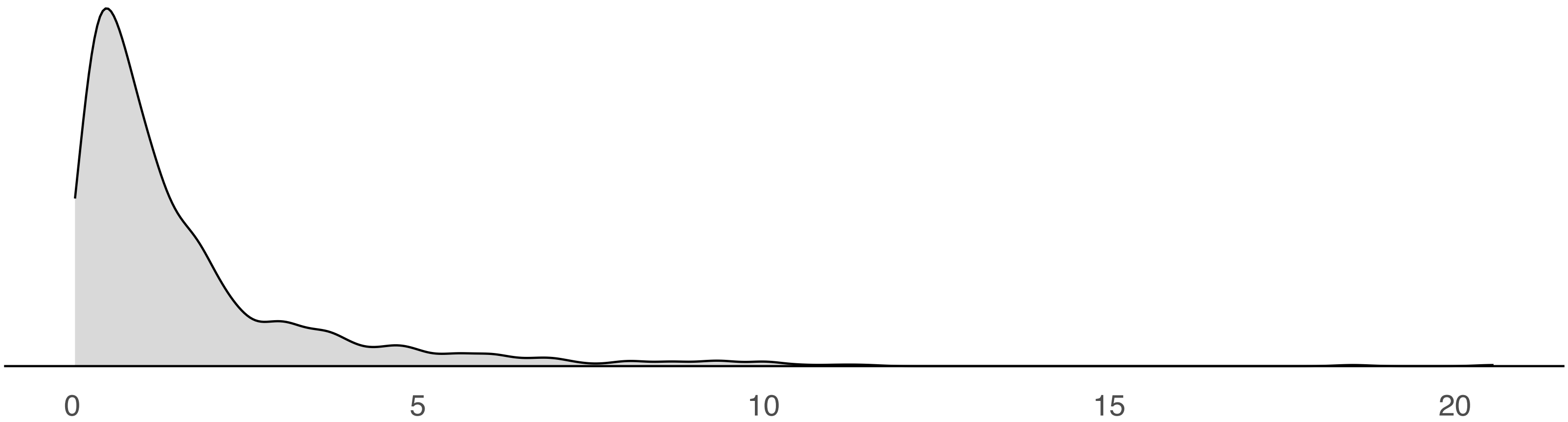
Key terms to describe p.d.f.'s:

1. A distribution can have **skew** (e.g., log-normal)
   - Skew means the distribution is asymmetric around its mean
2. A distribution can have a long **right tail** or **left tail** (e.g., fat-tailed climate sensitivity distributions!)
   - Long tails is a general term implying there is a lot of mass far away from the mean (not a precise defn.)
3. A distribution can be **symmetric**
   - The distribution is symmetric around its mean (Q: what does this imply about skew?)
4. A distribution can be **unimodal**, **bimodal**, or **multimodal**
   - A distribution with one (unimodal), two (bimodal), or more (multimodal) "peaks"

Shapes of probability distributions

Skew with a long right tail

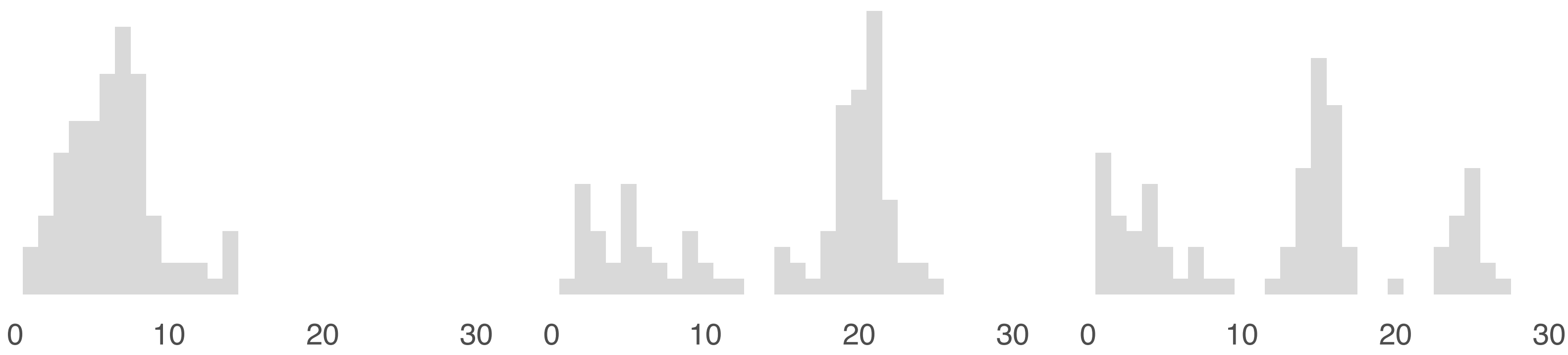
(log-normal sample distribution)



Shapes of probability distributions

Uni-, bi-, and multi-modal

(How many "peaks" do you see?)



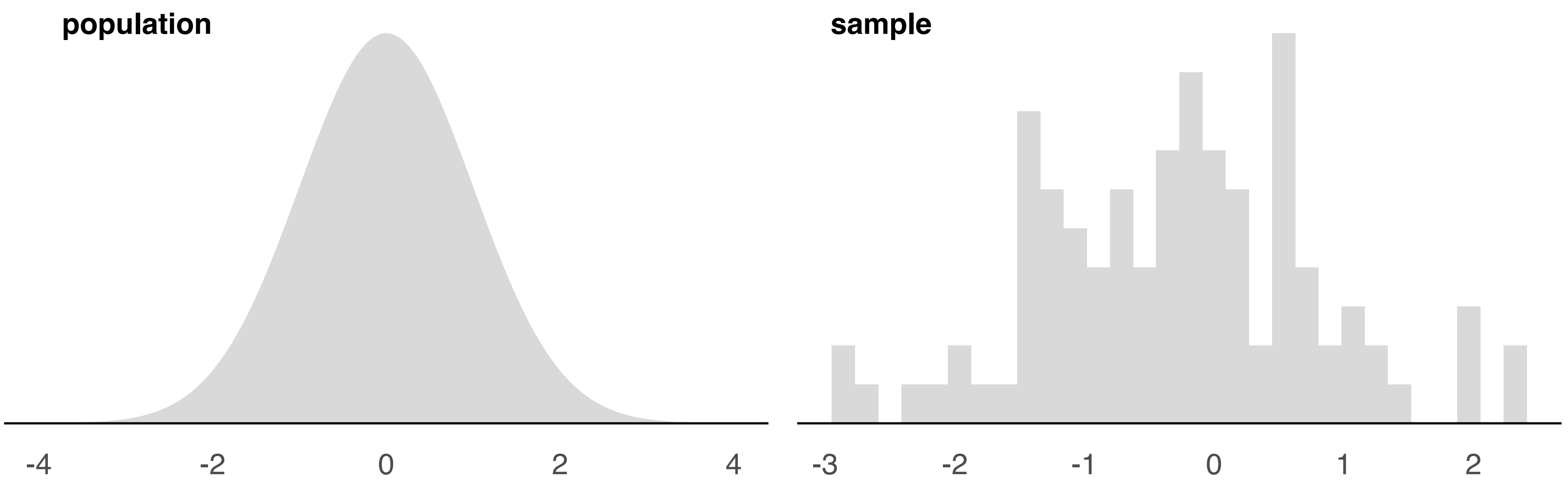
Summary statistics

Describing random variables

A probability density function describes a **population**

As we learned last week, we rarely have a **census** so we rarely can directly describe the p.d.f. itself.

Instead, we use **statistics** from a *sample* to estimate **parameters** of the *population*. Randomness in sampling means we call the variables in our sample "random variables"



Measures of central tendency

We often begin to describe a distribution using measures of **central tendency** (i.e., measures of the "middle").

Three are most common:

1. **Mean**
2. **Median**
3. **Mode**

Mean = expected value = average

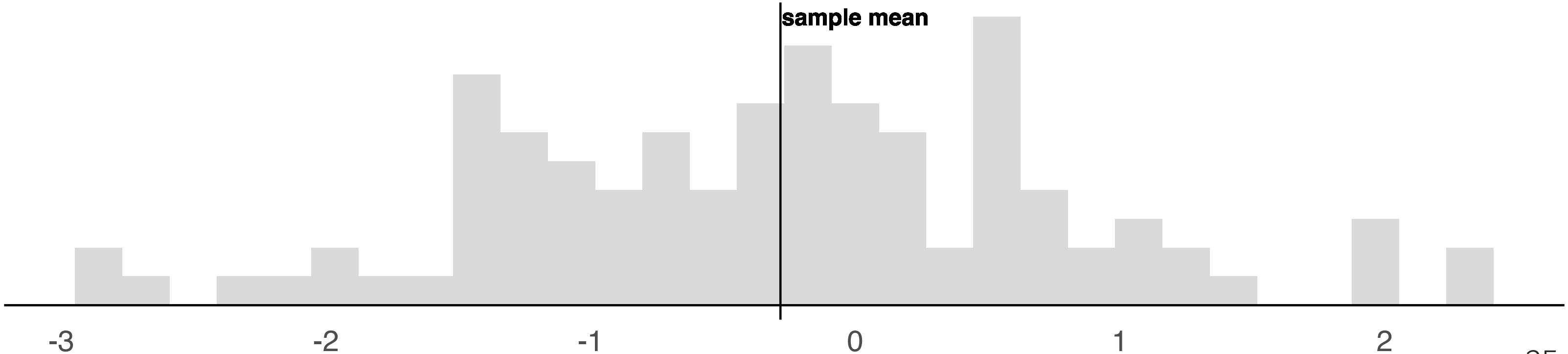
In a **population**, the mean is defined as:

$$\mathrm{E}[X] = \mu = \int_S x f(x) dx$$

In our **sample**, we compute the mean as:

$$\bar{x} = \frac{1}{n} \sum_{i \in n} x_i$$

We use $\bar{x}$ as an *estimate* of the parameter of interest, μ.



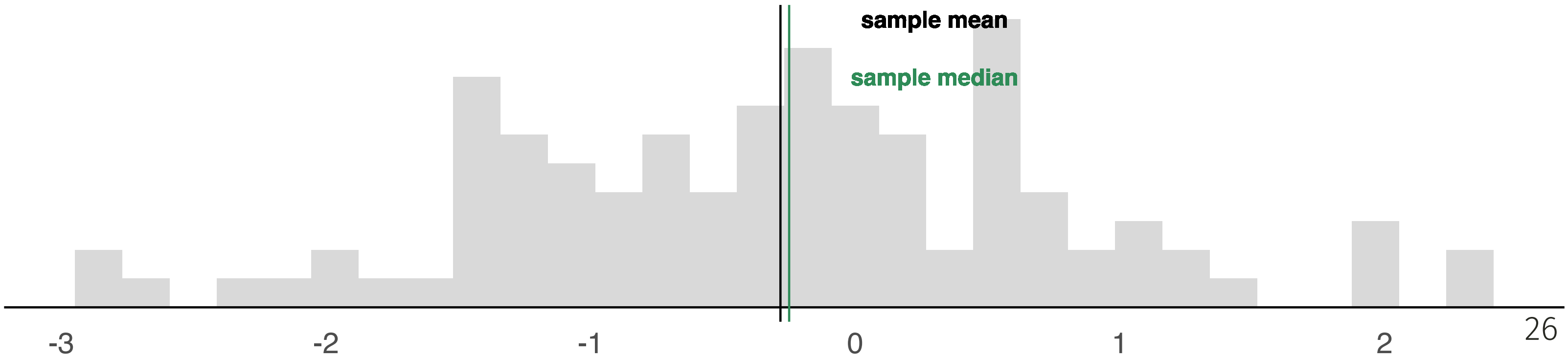
Median = middle value

In a **population**, the median is defined as the value m for which half the distribution falls below m and half above m:

$$P(X \leq m) = \int_{-\infty}^{m} f(x)dx = \frac{1}{2} = \int_{m}^{\infty} f(x)dx = P(X \geq m)$$

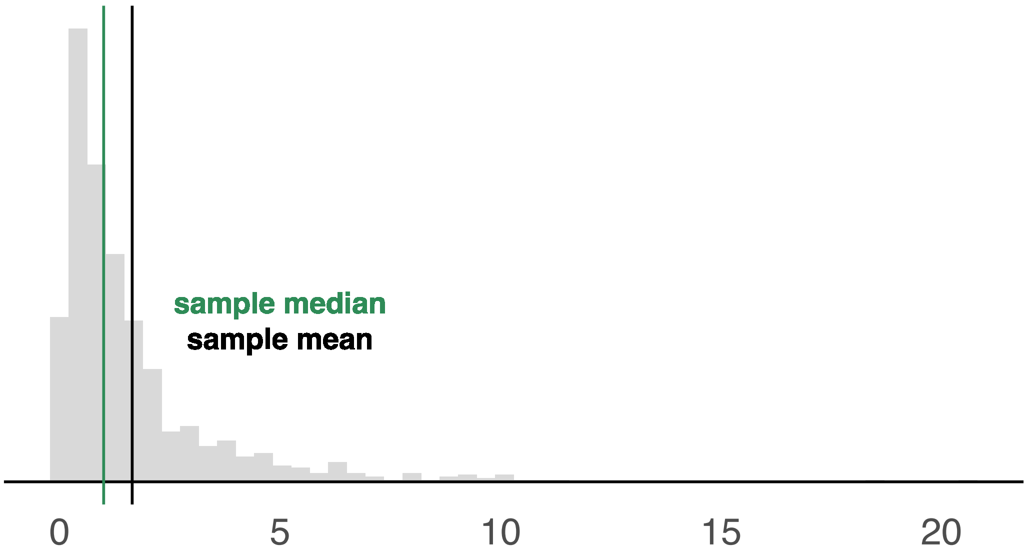
In our **sample**, we order all our data from lowest to highest and then compute the median as:

- n even? median = mean of the middle two values
- n odd? median = middle value



Median and mean are not always close

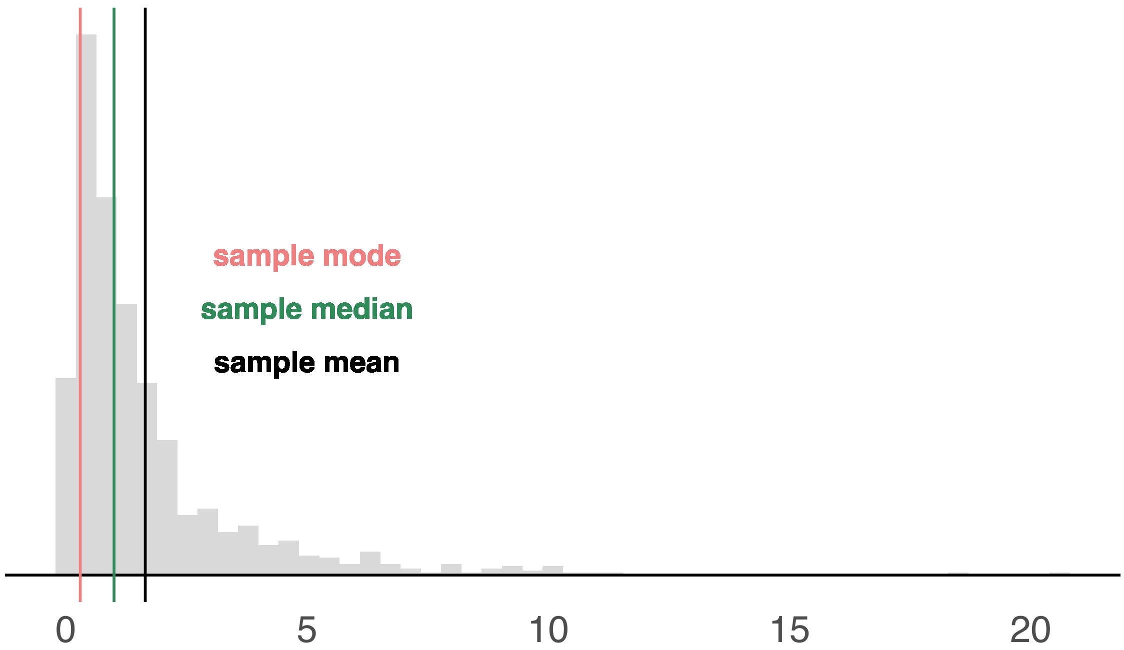
Non-normal distribution $\implies$ median and mean can diverge substantially



Mode = most frequent value

The **mode** is simply the most frequently observed value

This is much more useful for discrete data (ask yourself why!)



Measures of spread

Central tendency only gets us so far...we also need measures of **spread**.

1. **Range** (easy: min to max of your data)
2. **Variance**
3. **Standard deviation**
4. **Quantiles**

Measures of spread: Variance

Answers the question, how far are observations from the mean, on average?

In the population:

$$Var(X) = \mathbf{E}[(X-\mu)^2] = \sigma^2 = \int_{\mathrm{S}} (x-\mu)^2 f(x) dx$$

In the sample:

$$s^2 = \frac{\sum_{i \in n} (x_i - \bar{x})^2}{n-1}$$

> Q: Why do we divide by $n-1$?
>
> A: Lots of math to prove it (see here), but trust me, s^2 will be a biased estimate of σ^2 if you divide by n!

Units of variance: units of the random variable, *squared*

Measures of spread: Standard deviation

Just the square root of the variance!

In the population:

$$SD(X) = \sqrt{\mathrm{E}[(X-\mu)^2]} = \sigma = \sqrt{\int_{\mathrm{S}} (x-\mu)^2 f(x) dx}$$

In the sample:

$$s = \sqrt{\frac{1}{n-1} \sum_{i \in n} (x_i - \bar{x})^2}$$

Units of standard deviation: units of the random variable

Some helpful rules

$$\mathrm{E}[aX + b] = a\mathrm{E}[X] + b$$

$$\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y]$$

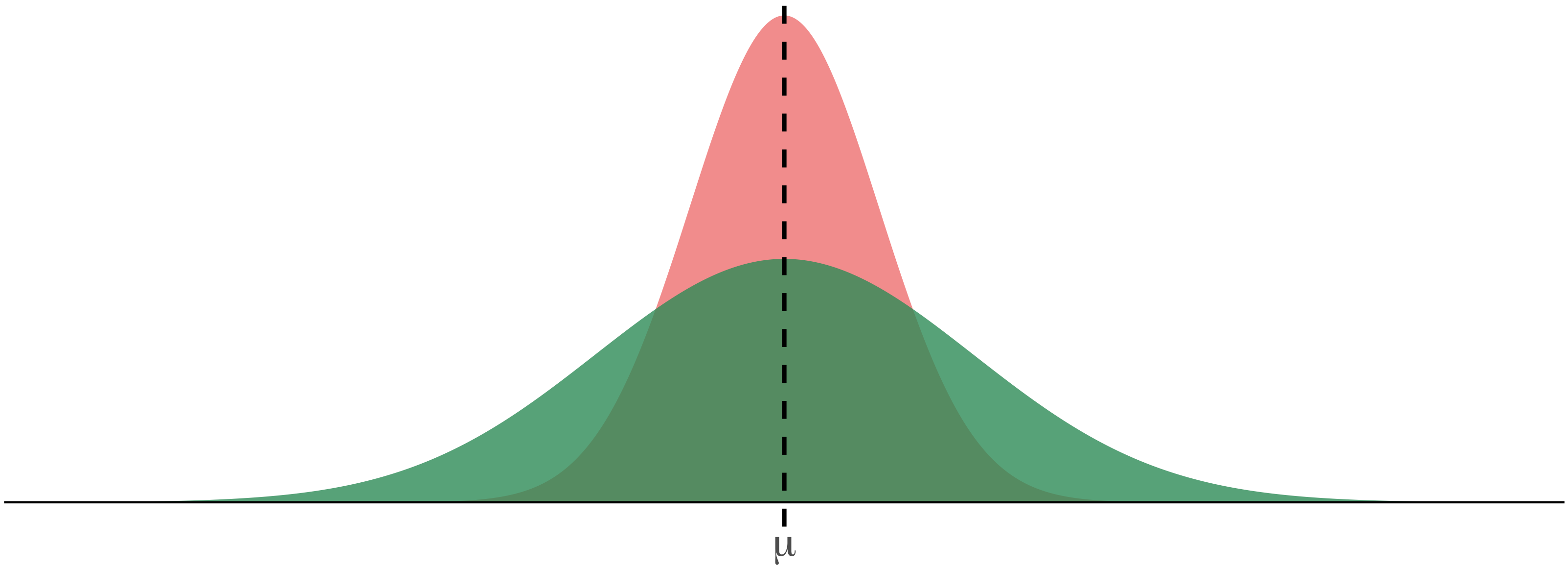
$$var(X) = \mathrm{E}[X^2] - (\mathrm{E}[X])^2$$

$$var(aX + b) = a^2 var(X)$$

Variance, visually

Pink: Low variance/standard deviation $\sigma = 1$

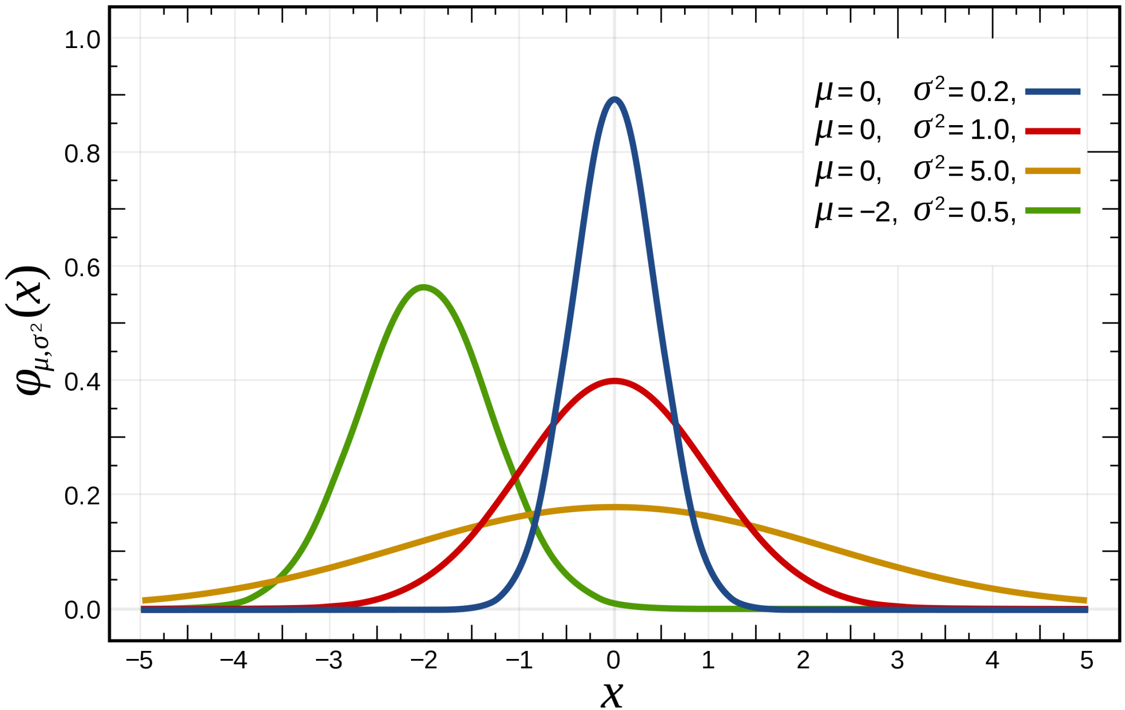
Green: High variance/standard deviation $\sigma = 2$



Variance, visually

Back to the normal distributions

- Changes in the *mean* shift the distribution right to left
- Changes in the *standard deviation* stretch the distribution out (or shrink it in)



Measures of spread: Quantiles

Quantiles are cut points of a probability distribution

In our sample, quantiles are cut points of our sample data

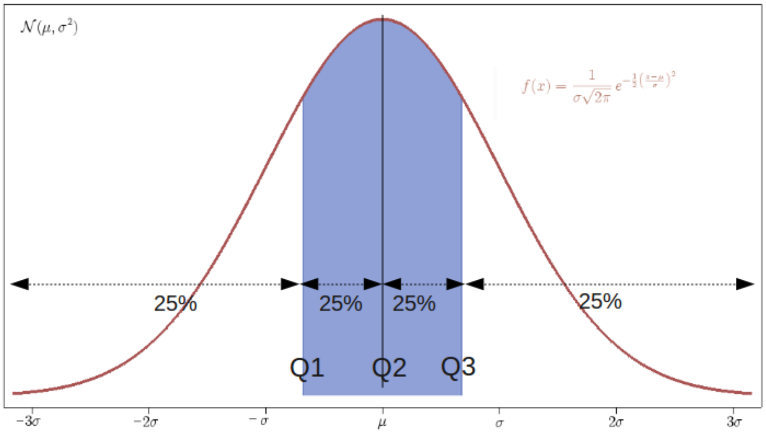
How do we compute them?

- We order our data from lowest to highest
- For the q-quantile, we divide these ordered data into q equal sized subsamples
- The value at the edge of the kth subsample is the kth q-quantile
  - This tells you the value below which $\frac{k}{q}$ of the data lie

> Question: How many q-quantiles are there for any given q?
>
> Answer: There are $q - 1$ of the q-quantiles

Example: The normal distribution

Common quantiles have names you have head of, such as *quartiles* for $q = 4$:



Quartiles of the normal distribution

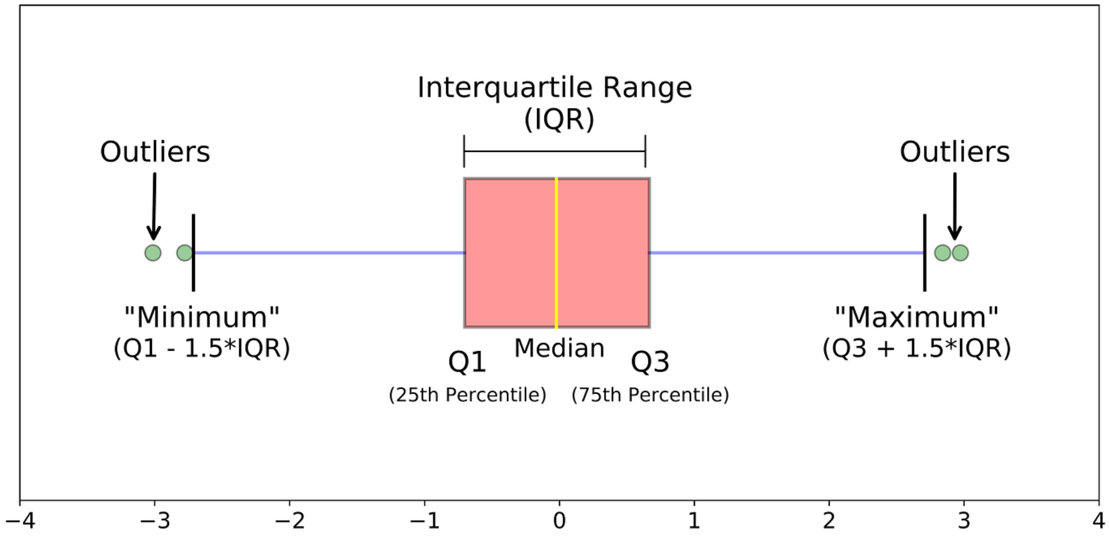
Interpretation: Q1 = first quartile, Q2 = second quartile, etc. The area below the red curve is the same below Q1 as it is between Q1 and Q2, between Q2 and Q3, and above Q3.

The Inter-quartile Range

The **inter-quartile range** (often called the IQR) is the 3rd quartile minus the 1st quartile (i.e., the range of the "middle" 50% of the data)

This is another measure of variability, like variance. Larger IQR = more variable data.

Often used as the edges of the box in a boxplot (we will do this in Lab!):



Common quantiles and interpretation

Common quantiles have names you have heard of:

- $q = 2$ **Median** tells us the value for which 50% of our sample sits *below* (and 50% above)
- $q = 3$ **Terciles**: tell us the values for which 33.33% (1st tercile) and 66.66% (2nd tercile) of our sample sits *below*
- $q = 4$ **Quartiles**: tell us the values for which 25% (1st quartile), 50% (2nd quartile), and 75% (3rd quartile) of our sample sits *below*
- $q = 10$ **Deciles**: tell us the values for which 10% (1st decile), ..., 50% (5th decile), ..., and 90% (9th decile) of our sample sits *below*

q The k*th* q-quantile tells us the value for which $\frac{k}{q} \times 100\%$ of our sample sits *below*

This sounds a lot like percentiles...

Percentiles are simply quantiles for q=100!

We hear about percentiles in daily life more often, and in practice people often use "percentiles" language for the more general term "quantiles".

Examples of percentiles:

- At 5'3", my height is the 40th percentile of the U.S. adult female height distribution $\rightarrow$ 40% of American female adults are shorter than me
- At 36 lbs, my son is the 90th percentile of U.S. male 3 year old weight distribution $\rightarrow$ 90% of American male 3 year olds are lighter than my son

> Exercise: Draw approximately where you think the 1st, 10th, 20th, 50th, 80th, 90th and 99th percentiles would be on a normal distribution.

Quantile-Quantile (Q-Q) Plots

Histograms plot the frequency of our data within bins

- `geom_histogram()` with `ggplot2` in `R`

Q-Q plots plot the quantiles of our data *against* quantiles of some theoretical distribution

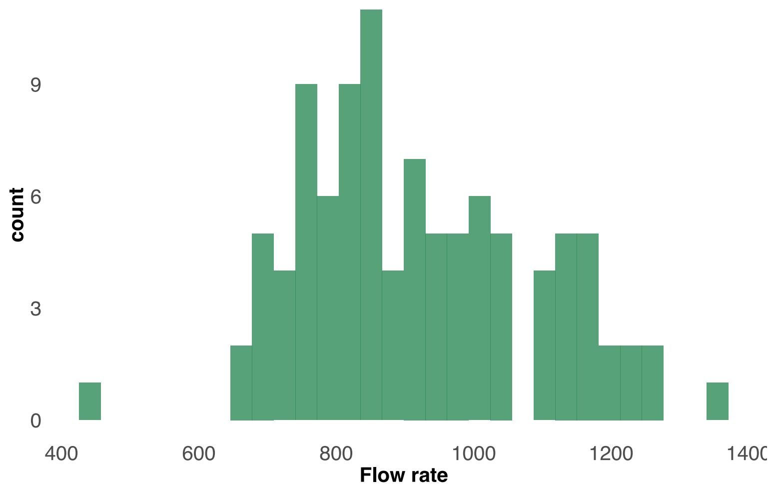
- `geom_qq()` with `ggplot2` in `R`

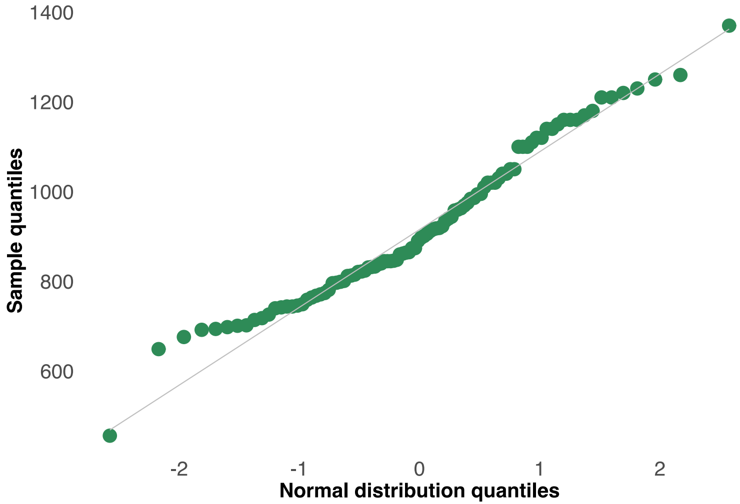
> This is helpful if we want to ask things like, are my data approximately normally distributed?

Straight line on a Q-Q plot indicates sample and theoretical distributions match

Q-Q plot: Example

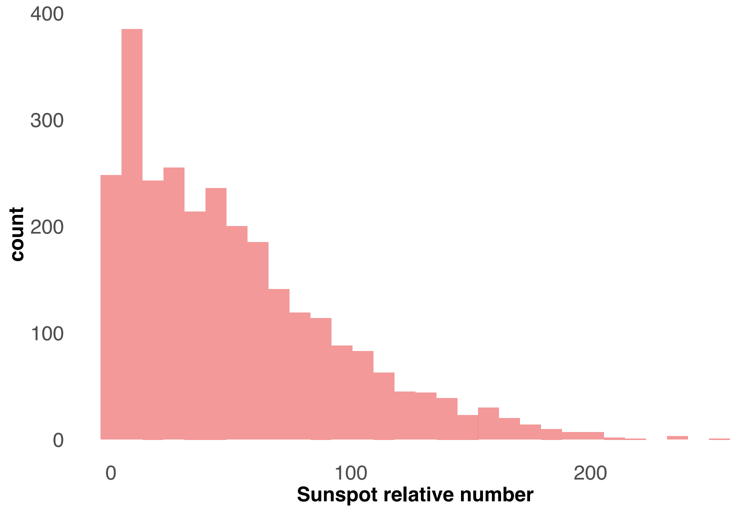
Annual flow of the river Nile at Aswan, 1871-1970, in 10^8 m^3

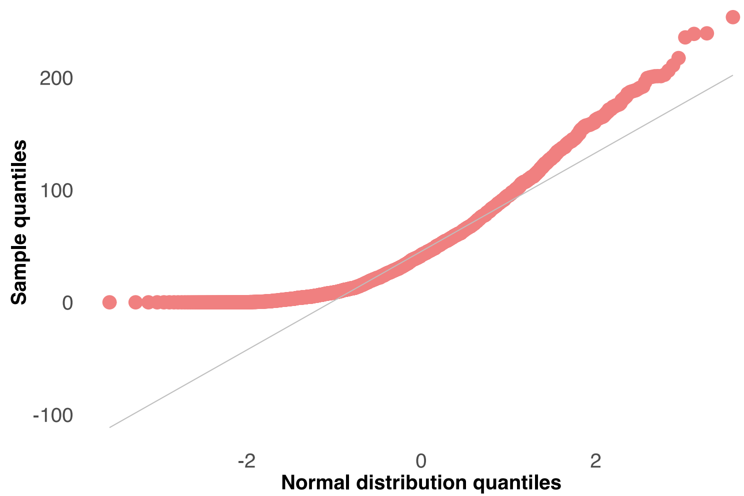




Q-Q plot: Example

Monthly mean relative sunspot numbers, 1749-1983

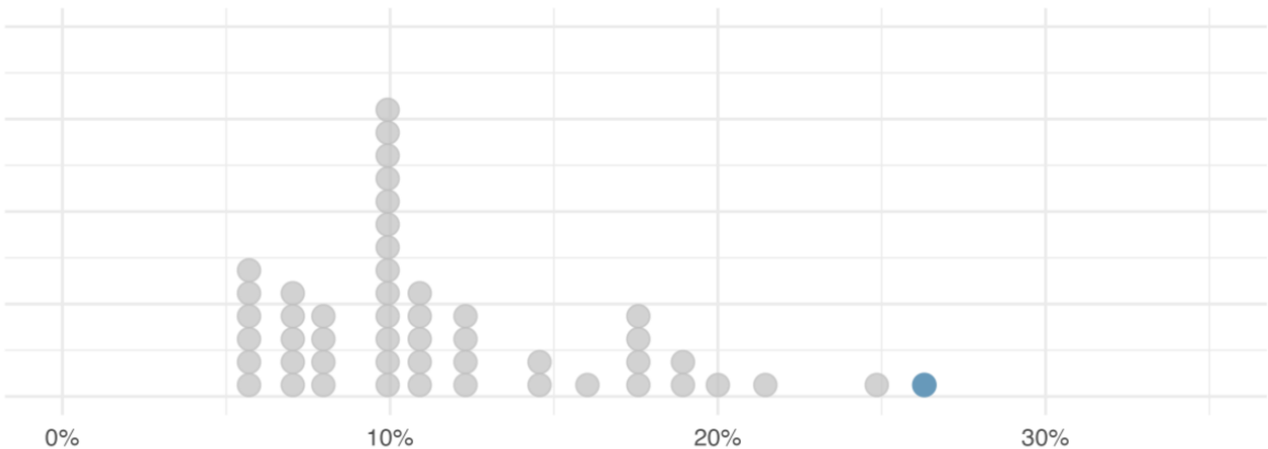




> We will continually return to the normal distribution. Always a good idea to check whether your data look normally distributed or not!

Which statistics are robust to outliers?

- Consider a sample of loans from a bank, each with an associated interest rate x.
  - $\bar{x} = 11.57$
  - $s = 5.05$
- The highest value in the data is somewhat of an outlier, $x_{max} = 26.3$.



Source: IMS, Ch. 5.6

Which statistics are robust to outliers?

- Consider a sample of loans from a bank, each with an associated interest rate.
    - $\bar{x} = 11.57$
    - $s = 5.05$
- The highest value in the data is somewhat of an outlier, $x_{max} = 26.3$.
- How do summary statistics change if we modify this outlier?

	Robust		Not robust	
Scenario	**Median**	**IQR**	**Mean**	**SD**
Original data	9.93	5.75	11.6	5.05
Move 26.3% to 15%	9.93	5.75	11.3	4.61
Move 26.3% to 35%	9.93	5.75	11.7	5.68

Table 5.4: A comparison of how the median, IQR, mean, and standard deviation change as the value of an extereme observation from the original interest data changes.

Law of large numbers

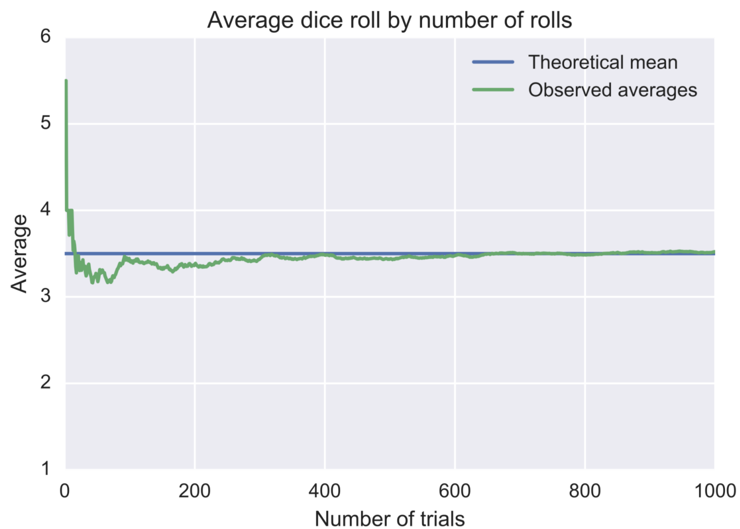
Big data

You probably have intuition that a larger sample is better than a smaller one...but why?

Suppose we have a **random** sample of some size n. How well does $\bar{x}$ approximate μ?

Law of large numbers:

$$\bar{x} \to \mu \text{ as } n \to \infty$$



Next up

Relationships between variables

Intro to ordinary least squares

Summarizing categorical and numerical data in `R` (Thursday lab)

Slides created via the R package **xaringan**.

Some slide components were borrowed from Ed Rubin's awesome course materials.